Torsion and Curvature

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Ever since Albert Einstein, in 1915, described gravity as the curvature of spacetime, physicists have been searching for a way to describe electromagnetism as some other geometric property of spacetime. Theodore Kaluza (1921) and Oscar Klein (1926) proposed the curvature of a 5-d spacetime to bring electromagnetism into the geometric picture. Contemporary string theory with its 10 and 26 dimensions of spacetime can be viewed as a long delayed (and very complicated) outgrowth of this hyperspace approach the the unification program; and in this case all the forces (including the two nuclear forces--strong and weak) are included in the unification. Here I will describe a different approach, which entails both curvature and torsion as physically meaningful quantities.

In the ordinary 3-d Euclidean space, we can tell if two vectors at distant points are parallel, by moving one vector without rotating it to coincide (at its base) with the other. However if a space is curved, it is impossible to compare two distant vectors without some method of parallel transport of vectors throughout the curved space. The amount of curvature is a measure of the mismatch of a vector with a copy of itself which has undergone a complete circuit. In order for this to be valid, the moving vector must be parallel transported, and any rotation of the vector must be due to the curvature encountered along the circuit. The parallel transport is provided by a structure which is added to the manifold and is called the connection.

In the theory of general relativity, the connection is provided by an object called the Christoffel symbol Γ^k_{ij}. This is a very compact notation for a set of 40 (= 64 -24) functions on the 4-d spacetime. If the symbol carried two asymmetric lower indices, there would be 64 (= 4 x 4 x 4) functions; but the symmetry of the lower indices reduces the independent functions to 40. The standard Christoffel symbol of general relativity is symmetric in the two lower indicies i,j, and generates a connection called the Levi-Civita connection. However, there are geometries for which an asymmetic Christoffel symbol is employed in addition to the the symmetric Christoffel symbol. The asymmetry is carried by a tensor T called the torsion. We can write:

$$\Gamma^k_{ij} - \Gamma^k_{ji} = T^k_{ij}$$

Thus although the Christoffel symbols are not tensors, their difference is a tensor. In physics, we expect tensors to correspond to measurable quantities. If T is 0, then the torsion is zero, and the symbol must be symmetric. A very special case of parallel transport is called absolute parallelism. While ordinary parallel transport guarantees that the vectors will be rotated only by the curvature along the particular path in the circuit, an absolute parallelism connection guarantees that the vectors will remain unrotated by travel along any circuit that follows vector field flow lines. This implies that there is no curvature for this absolute parallelism connection. However their will, in general, be a gap in this circuit caused by a "vertical" motion of the the moving vector. After making the ciruit, the moving vector and its stay-at-home twin will, end up parallel to each other but separated by this "vertical" gap. This gap is called the torsion.

It is strange that the concept of parallel transport of vectors was introduced by Tullio Levi-Civita and J.A. Schouten, independently of each other, in 1917, two years after Einstein's general relativity theory was published. This was, in effect, a reinterpretation of the role of the Christoffel symbol, which was used by Einstein. Not until 1922 did Elie Cartan (and Weitzenbrock [Ref. 1]) come up with the idea of absolute parallelism. It was an idea he associated with an attempt to describe electromagnetism as the torsion of a connection structure with absolute parallelism. After Einstein independently arrived at the same idea (which he called *Fernparallelismus* or distant parallelism) in 1928, Cartan wrote to Einstein, and a series of letters ensued which lasted until 1932. In his first letter of 8 May 1929, Cartan reminded Einstein of a very simple example of absolute parallelism [2]:

"I even remember trying, at Mr Hadamard's home, to give you the simplest example of a of a Riemannian space with *Fernparallelismus* by regarding two vectors within a sphere making the same angle with the meridian lines passing through their origins as parallel: the corresponding geodesics are the rhum lines."

This is an example of absolute parallelism with torsion, since rhum lines (or loxodromes) as straight lines correspond to the Mercater mapping of the Earth's surface. (Notice that only the meridians and the equater of the Mercater map are geodesics on the globe; and all other great circle routes are not Mercater straight lines. Thus Cartan's use of "geodesic" is ambiguous; he should have said: "Instead of geodesics we have rhum lines.") Mercater maps are useful for navigation because a rhum line corresponds to a constant compass bearing (or azimuth). As is vividly obvious, the Mercator maps distort distances toward the polar regions, so that, for example, Greenland is depicted as being larger than South America. This stretching along the east -west parallels is a distortion which in this context is called torsion.

There is a caveat in using the two-sphere S^2 as an example of absolute parallelism; for as we will see later S^2 cannot be a manifold of absolute parallelism because it is not a group manifold. In Cartan's example (where S^2 is the surface of the earth), it is actually the magnetic dipole field keeping magnetic compasses pointing north which provides the absolute parallelism. It may have been this very fact that suggested to Cartan the idea that torsion in the more appropriate setting of a group manifold could be used to descibe electromagentism. As Cartan put it in his first letter to Einstein [2]:

"I have systematically studied the tensors which arise from either the curvature or the torsion: one of those given by the torsion has precisely all the mathematical characteristics of the electromagnetic potential."

Einstein's problem is that he wants both curvature and torsion in his geometry: curvature for gravity, torsion for electromagnetism. The connection structure which provides curvature, is based on the symmetric Christoffel symbol. Thus this connection (called the Levi-Civita connection) has zero torsion. By contrast, the absolute parallelism connection which provides torsion has zero curvature.

Cartan, who had recently introduced the concept of a connection into geometry never used this word in his letters to Einstein. With hindsight, we can see that the problem of describing a geometry with both curvature and torsion would have been clarified by the use of the connection formalism.

Since curvature and torsion are relative to the connection, the problem of describing a geometry with both curvature and torsion becomes that of describing the relationship between two connections. That is: what is the relationship between the Levi-Civita connection (with curvature but zero torsion) and the Cartan connection (with torsion but zero curvature)? As Cartan showed in several papers of the 1923-1927 period, there are good examples of spaces carrying both these connections. These spaces are Lie group manifolds. In fact, later work by Joseph Wolf proved that the only spaces that carry an absolute parallelism (Cartan) connection are Lie groups--with one exception: the seven-sphere S^7.

Well known to topologists is the fact that the only spheres that carry an absolute parallelism are spheres of dimension 1, 3, and 7. And the only spheres that are Lie groups are spheres of dimensions 1 & 3. The Lie group structures of these spheres are called U(1) and SU(2). Moreover, S^1 (= U(1)) is the set of all unit complex numbers, while S^3 (= SU(2)) is the set of all unit quaternions, and S^7 is the set of all unit octonions (or Cayley numbers); it is because octonions are not an associative algebra that S^7 fails to be a Lie group; but the octonion structure provides an absolute parallelism on S^7. The fact the the two-sphere S^2 fails to be a Lie group, and thus has no absolute parallelism, corresponds the fact that the “hair on a two-sphere cannot be combed.” By contrast, the hair on a two-torus can be combed. This is, of course, because a torus (of any dimension) is a Lie group (in fact a commutative Lie group) and thus has an absolute parallelism connection.

To see this connection structure on an ordinary torus, imagine hair growing on the torus and then imagine running your hand around the torus smoothing all the hairs (as analogs of vectors) to go along a flow around the torus. According to this connection structure (provided by the smoothed out vector field) there is zero curvature; and we say that a torus is flat. There is, however, no torsion either because the torus Lie Group is commutative.

To explain why commutativity negates torsion in a Lie group manifold, we must describe the structure of absolute parallelism in a Lie group. On any manifold an infinity of different tangent vector fields can live. In general, the set of all tangent vector fields on a manifold constitute a finite dimensional vector space which is also an associative algebra, and which can be made into a Lie algebra by the Lie bracket operation on the vector fields: [X,Y] = XY -YX . However, there is no concept of invariance for these vector fields unless the underlying manifold is a Lie group manifold. A Lie group is a manifold all of whose elements (or points) are also elements of a group. Elements of a group are represented by matrices. Matrices act on vectors, by changing one vector into another. Thus groups act on vector spaces of different dimension corresponding to the various representation matrices of the group. Of course, groups also act on themselves; and this action can be represented by the multiplication of one matrix by another matrix, so that this action is (in general) non-commutative. In particular there is a so-called adjoint representation, in which the dimension of the matrix is the same as the dimensionality of the group manifold. The importance of this adjoint representation, is that in this way the group acts on the vector space consisting of the Lie algebra of the Lie group. Now the Lie algebra can be described as the set of all left-invariant vector fields (or equivalently the right-invariant vector fields) which live on the Lie group manifold.

Most importantly for the physics of torsion, it is the left-invariance (or right invariance) of the Lie algebra vector fields the provides absolute parallelism. As Cartan discovered [3], there are three canonical connections on a Lie group manifold. These three connections are generated by three different actions of the Lie group on itself:

(1) Left action: g --> $h\,g$ (where g and h are group elements of Lie group G)

(2) Right action: g --> $g\,h$ "

(3) Adjoint action: g --> $h^{-1}gh$ (where h^{-1} is the inverse element of h)

The left (and right) actions transport tangent vectors in an absolute parallel maner from any point on a geodesic flow line (called an integral curve) of a vector field to any other point of the same vector field (but on a different flow line)--thus identifying the tangent vectors of these flow lines as parallel. Since this absolute-parallel transport works for any vector field on G, the transport between point a to point b (both of which are group elements) is a transport of the tangent plane at a to the tangent plane at b. It is these absolute-parallel transports that provide the one-to-one correspondence between the two definitions of the Lie algebra as: (1) the set of left (or right)-invariant vector fields: and (2) the tangent plane at the identity element of the Lie group manifold. A single vector of this tangent plane by the group action (left or right):

e --> ge = g or e --> eg = g

(where e is the identity element) carries that vector from e to g. Thus the entire Lie group manifold is covered by copies of that single vector; and this field of vectors is a left (or right) invariant vector field. And so there is a one-to-one correspondence between any tangent vector at the identity and some particular vector field, which is left (or right) invariant by construction. If we similarly set up all the left (or right) invariant vector fields, we will have a copy of the identity tangent plane at each point of the Lie group manifold. The set of all these tangent planes together form a vector bundle called the tangent bundle of the Lie group. For the Lie group G, the symbol for the tangent bundle is TG, and it is simply the direct product of the Lie group G and the Lie algebra *g*. [Note the standard convention of lower-case letters for Lie algebras--not to be confused with the lower-case letters used above to depict Lie group elements.]
We can write:

$$TG = g \quad G$$

Thus a tangent vector field is a cross section of TG. This means that each element of G has a copy of the Lie algebra *g* attached to it as a fiber. Any manifold M has a tangent bundle TM where each fiber is the set of tangent vectors at some point of M. However, we cannot in general write TM as a direct product, although in any local region of M, there is a direct product structure, which is called a local trivialization of the bundle structure. In contrast to the case of an ordinary manifold, which is not a Lie group, we say that TG is a trivial bundle because it is direct product of the base space G with the the fiber *g*, this implies a global trivialization of the bundle structure; moreover, this global trivialization corresponds to the absolute parallelism afforded by the group action on the group manifold and thus on the parallel transport of vectors of the Lie algebra, as described above.

The intimate relationship between the Lie group G and the Lie algebra *g*, has the consequence that the torsion of G afforded by *g*, is simply the Lie product, [x,y], of *g* [Ref. 3, 4].

Thus if the Lie product is zero, the product is commutative and the torsion is zero. In general, for the torsion T of a Lie group manifold we can write:

$$[X_i, Y_j] = T_{ij}^k Z_k \qquad \text{(T is usually written as C.)}$$

where the componets of T are the structure constants of the Lie algebra; and X, Y, and Z are Lie algebra elements, i.e., left-invariant vector fields on G. For right invariant vector fields the torsion tensor is the would be $-T_{ij}^k$. Note that the Lie algebra structure-constant tensor is usually called C_{ij}^k, and that although vector fields on any manifold form a Lie algebra, only if this manifold is a Lie Group, are the structure constants torsional, so that T = C, for left-invariant vector fields on the Lie Group manifold.

Since the Lie algebra has the same dimensionality as the Lie group, for an n-dimensional Lie group, there would be n basis elements of the Lie algebra each of which corresponds to an independent vector field, i.e., a field of tangent vectors to a congruance of geodesic flow lines on the Lie group manifold .

These geodesic flow lines are a third parallel transport structure on the Lie group manifold. In this case, however, the parallel transport is not absolute parallelism. The geodesic flow lines of the vector fields parallel transport their own tangent vectors, and any other tangent vector attached to a flow line will be rotated only be the curvature encountered by the path. Thus if a tangent vector is transported around a closed path (along a quadrilateral of geodesic flow lines), the transported tangent vector can be compared with a copy of itself left behind at the starting point of the closed circuit. Any curvature in the manifold inside the circuit will show up as a mismatch in direction between the two tangent vectors. In general, the curvature tensor describing the curvature of the Lie group manifold is the Riemann curvature tensor which can be written in terms of the Lie algebra structure constants [Ref. 3, pp 188, 217.]:

$$R_{i,kl}^k = \frac{1}{4} C_{hi}^j C_{kl}^h$$

The Riemann curvature tensor is the tensor generated by the Levi-Civita connection, for which there is zero torsion.

Thus on the Lie group manifolds we have two radically different connections: the Cartan asymmetric connection which has torsion but no curvature, and the symmetric Levi-Civita connection which has curvature but no torsion.

Given two connections on the same manifold, the difference between the Christoffel symbols is a tensor, called the difference tensor. In the case of these two connections on a Lie group manifold the difference tensor is the contorsion tensor K. We can write:

$$K_{ij}^{k} = \Gamma_{ij}^{k} - \overline{\Gamma}_{ij}^{k}$$

where the first Christoffel symbol is the Cartan connection of absolute parallelism (with 64 independent functions); the second (overbarred) Christoffel symbol is the Levi-Civita connection (with 40 independent functions, because of the symmetry of the i,j $A_n: o - o - \ldots - o$indices); and K is the contorsion tensor (with 24 independent functions). [Ref. 5,6]

The contorsion tensor is also the tensor of Ricci rotation coefficients. So now we can compare the Lie group connection structure with the formalism of Gennady Shipov [Ref. 7]. His space of absolute parallelism is the 10-d manifold of the Poincare group. It is the non-commutative part of this group, SO(3,1) that provides the torsion. The double cover of this group is SL(2,C), which provides the spinor rotational coefficients analogous to the Ricci rotational coefficients.

Since SL(2,C) corresponds to the A_1 graph (with one node, depicted as o) in the heirarchy of A-D-E Coxeter graphs, which classify a host of mathematical objects [8]:

A_n: o--o--o...-o

D_n: o--o--o...-o
          |
          o

E_6: o--o--o--o--o
             |
             o

E_7: o--o--o--o--o--o
             |
             o

E_8: o--o--o--o--o--o--o
             |
             o

I believe that the entire A-D-E hierarchy in all its classifications of hyperspace structures: i.e., reflection groups, crystallographic lattices, Lie algebras (and groups), Thom catastrophes, Arnold singularities, caustics, gravitational instantons, etc., will contribute to higher levels of curvature, torsion, and many other properties [Ref. 9, 10].

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